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Acoustic Diffraction by a Rigid Annular Disk

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SUMMARY
The subject of this paper is the problem of acoustic diffraction by a perfectly rigid annular disk. The method of solution rests on formulating the problem in terms of an integral equation which embodies the steady state wave equation as well as the boundary conditions. This Fredholm integral equation of the first kind is converted into four simultaneous integral equations of the second kind by using Williams' integral equation technique. These four integral equations are subsequently solved by the standard iterative procedure when the frequency of the incident wave is low and the inner radius of the annulus is small.

1. Introduction

Recently a great interest has been shown in three-part boundary-value problems in mathematical physics and theoretical mechanics [1–12]. In the field of diffraction, the problem of acoustic diffraction by a soft annular disk has been solved by Thomas [10], while the corresponding problem of electromagnetic diffraction by a perfectly conducting annular disk has been solved by the present authors [12]. The problem of acoustic diffraction by a perfectly rigid annular disk is also of interest and forms the subject of the present paper.

The method of solution rests on formulating the problem in terms of an integral equation which embodies the steady state wave equation as well as the boundary conditions. This integral equation, which is a Fredholm integral equation of the first kind, is converted into four simultaneous integral equations of the second kind by using Williams' integral equation technique [8] as illustrated by Thomas [10]. These four integral equations are subsequently solved by the standard iterative procedure when the frequency of the incident wave is low and the inner radius of the annulus is small.

The formulation of the problem as well as the solution is given for a general acoustic wave. A detailed discussion is then presented for the special case when the incident wave is an axially symmetric plane wave. The values of the far field amplitude as well as the scattering cross-section are presented. A complete study is made both for inner as well as outer edge conditions. The results so obtained are subsequently verified for the special case mentioned above. When the inner radius of the annular disk tends to zero, the results of this paper reduce to the known results for a perfectly rigid circular disk.

2. Formulation of the Problem

We take cylindrical polar coordinates (ρ, φ, z), with z-axis along the axis of the annular disk such that the annular disk is defined by

$$z=0, \quad b \leqq \rho \leqq a, \quad \text{all } \varphi .$$

Let the time dependence be $\exp(-i\omega t)$ and let the time-independent part of the velocity potentials of the incident and diffracted fields be $u_0(\rho, \varphi, z)$ and $\Phi(\rho, \varphi, z)$. Both these functions satisfy the Helmholtz equation. The total velocity potential u is

$$u(\rho, \varphi, z) = u_0(\rho, \varphi, z) + \Phi(\rho, \varphi, z) ,$$

and we have to solve the following boundary-value problem

$$(\nabla^2+k^2)\Phi=0\,, \tag{1}$$

$$\frac{\partial\Phi}{\partial z}=-\frac{\partial u_0}{\partial z} \quad \text{on} \quad z=0\,, \quad b\leqq\rho\leqq a\,, \tag{2}$$

Φ and $\dfrac{\partial\Phi}{\partial z}$ are continuous across $z=0$, $0\leqq\rho<b$, $\rho>a$, (3)

where $k=\omega/c$ is the wave number and c is the speed of the wave propagation. In addition $\Phi(\rho,\varphi,z)$ satisfies the Sommerfeld radiation condition as well as appropriate edge conditions.

The Fredholm integral equation of the first kind which embodies the boundary value problem (1) to (3) is

$$\left[\frac{\partial u_0(\rho,\varphi,z)}{\partial z}\right]_{z=0}=-\frac{1}{4\pi}\int_b^a\int_0^{2\pi} tI(\boldsymbol{r}_1)\left[\frac{\partial^2}{\partial z^2}\left(\frac{e^{ikR}}{R}\right)\right]_{\substack{z_1=0\\ z=0}} d\varphi_1\,dt\,, \quad (b\leqq\rho\leqq a)\,, \tag{4}$$

where

$\boldsymbol{r}_1=(t,\varphi_1,z_1)$ is a point on the annulus,

$$R=\{\rho^2+t^2-2\rho t\cos(\varphi-\varphi_1)+(z-z_1)^2\}^{\frac{1}{2}}\,,$$

and

$$[I(\boldsymbol{r}_1)]_{z_1=0}=\{[\Phi(t,\varphi_1,z_1)]_{z_1=0^-}-[\Phi(t,\varphi_1,z_1)]_{z_1=0^+}\}\,,$$

is the jump in the potential Φ across the annulus. We now assume that $u_0(\rho,\varphi,z)$ has the form $u_0^{(m)}(\rho,z)\cos m\varphi$ with m an arbitrary positive integer or zero. More general excitation can be formed from a superposition of such modes. By writing

$$I(t,\varphi_1,0)=2g^{(m)}(t)\cos m\varphi_1\,,$$

the equation (4) becomes

$$\cos m\varphi\left[\frac{\partial u_0^{(m)}(\rho,z)}{\partial z}\right]_{z=0}=-\frac{1}{2\pi}\int_b^a tg^{(m)}(t)\int_0^{2\pi}\frac{\partial^2}{\partial z^2}\left[\frac{e^{ikR}}{R}\right]_{\substack{z_1=0\\ z=0}}\cos m\varphi_1\,d\varphi_1\,dt\,,$$

$$(b\leqq\rho\leqq a)\,. \tag{5}$$

Following Williams [13], the integral equation (5) can be reduced to

$$\left[\frac{\partial u_0^{(m)}(\rho,z)}{\partial z}\right]_{z=0}=\left[\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}+k^2\right)\int_b^a tg^{(m)}(t)K^{(m)}(t,\rho)dt\right]_{z=0}\,,$$

where $(b\leqq\rho\leqq a)$. (6)

$$K^{(m)}(t,\rho)=\int_0^{2\pi}\frac{\exp\{ik(\rho^2+t^2-2\rho t\cos\psi+z^2)^{\frac{1}{2}}\}}{2\pi(\rho^2+t^2-2\rho t\cos\psi+z^2)^{\frac{1}{2}}}\cos m\psi\,d\psi$$

$$=\int_0^\infty\frac{p\,e^{-\gamma|z|}J_m(p\rho)J_m(pt)dp}{\gamma}\,, \tag{7}$$

$$\gamma=\begin{cases}(p^2-k^2)^{\frac{1}{2}} & p\geqq k\,,\\ -i(k^2-p^2)^{\frac{1}{2}} & k\geqq p\,,\end{cases}$$

and J_m is Bessel function of order m.

Next step is to use the formulae

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\right)J_m(p\rho)=-p^2J_m(p\rho)\,,$$

$$J_m(px)=\frac{1}{px^{m+1}}\frac{d}{dx}[x^{m+1}J_{m+1}(px)]\,,$$

and the edge conditions $g^{(m)}(a)=g^{(m)}(b)=0$. Then from (6) and (7) we get, after integrating by parts and putting $z=0$, the result

$$\rho^{m+1}\left[\frac{\partial u_0^{(m)}(\rho, z)}{\partial z}\right]_{z=0} = \frac{\partial}{\partial\rho}\left[\rho^{m+1}\int_b^a\int_0^\infty \gamma\frac{t^{m+1}}{p}\frac{d}{dt}\right][t^{-m}g^{(m)}(t)]\,J_{m+1}(p\rho)J_{m+1}(pt)\,dp\,dt\,,$$
$$(b\leqq\rho\leqq a)\,. \quad (8)$$

Now set

$$f^{(m)}(\rho) = \frac{d}{d\rho}[\rho^{-m}g^{(m)}(\rho)]\,, \qquad b\leqq\rho\leqq a\,, \tag{9}$$

in (8) and integrate to get the integral equation

$$\int_b^a\int_0^\infty \frac{t^{m+1}}{p} f^{(m)}(t)\gamma J_{m+1}(p\rho)\,J_{m+1}(pt)\,dp\,dt$$
$$= \frac{1}{\rho^{m+1}}\int_0^\rho t^{m+1}u_0^{(m)\prime}(t)\,dt\,, \qquad (b\leqq\rho\leqq a)\,, \tag{10}$$

where $u_0^{(m)\prime}(\rho)$ denotes $\left\{\dfrac{\partial u_0^{(m)}(\rho, z)}{\partial z}\right\}_{z=0}$.

This is an integral equation of the first kind in unknown function $f^{(m)}(t)$. Our aim is to convert it to a few simultaneous integral equations of the second kind so that we can use the iteration scheme.

By writing

$$\frac{\gamma}{p}J_{m+1}(p\rho)\,J_{m+1}(pt) = J_{m+1}(p\rho)\,J_{m+1}(pt) + \left(\frac{\gamma}{p}-1\right)J_{m+1}(p\rho)J_{m+1}(pt)\,,$$

in the equation (10), we have

$$\int_b^a\int_0^\infty t^{m+1}f^{(m)}(t)\,J_{m+1}(p\rho)\,J_{m+1}(pt)\,dp\,dt$$
$$= \frac{1}{\rho^{m+1}}\int_0^\rho t^{m+1}u_0^{(m)\prime}(t)\,dt - \int_b^a\int_0^\infty t^{m+1}f^{(m)}(t)\left(\frac{\gamma}{p}-1\right)\times$$
$$\times J_{m+1}(p\rho)J_{m+1}(pt)\,dp\,dt\,, \qquad (b\leqq\rho\leqq a)\,. \tag{11}$$

This method rests on setting

$$\frac{1}{\rho^{m+1}}\int_0^\rho t^{m+1}u_0^{(m)\prime}(t)\,dt = \sum_{r=-\infty}^{\infty} a_r^{(m)}\rho^r\,, \qquad (b\leqq\rho\leqq a)\,, \tag{12}$$

and introducing two functions $h_1^{(m)}(\rho)$ and $h_2^{(m)}(\rho)$ such that

$$h_1^{(m)}(\rho) = \sum_{r=0}^{\infty} a_r^{(m)}\rho^r\,, \qquad (0\leqq\rho< a)\,, \tag{13}$$

and

$$h_2^{(m)}(\rho) = \sum_{r=-\infty}^{-1} a_r^{(m)}\rho^r\,, \qquad (b<\rho<\infty)\,. \tag{14}$$

Thus equation (11) splits into the following two equations:

$$\int_0^\infty t^{m+1}f_1^{(m)}(t)\int_0^\infty J_{m+1}(p\rho)\,J_{m+1}(pt)\,dp\,dt$$
$$= h_1^{(m)}(\rho) - \int_0^\infty t^{m+1}f_1^{(m)}(t)\int_0^\infty\left(\frac{\gamma}{p}-1\right)J_{m+1}(p\rho)\,J_{m+1}(pt)\,dp\,dt\,, \qquad (0<\rho<a)\,, \tag{15}$$

and $\int_0^\infty t^{m+1} f_2^{(m)}(t) \int_0^\infty J_{m+1}(p\rho) J_{m+1}(pt)\, dp\, dt$

$$= h_2^{(m)}(\rho) - \int_0^\infty t^{m+1} f_2^{(m)}(t) \int_0^\infty \left(\frac{\gamma}{p} - 1\right) J_{m+1}(p\rho) J_{m+1}(pt)\, dp\, dt\,, \qquad (b<\rho<\infty), \quad (16)$$

where

$$f_1^{(m)}(\rho)+f_2^{(m)}(\rho) = \begin{cases} 0, & 0 \leqq \rho < b\,, \\ f^{(m)}(\rho), & b \leqq \rho \leqq a\,, \\ 0, & a < \rho < \infty\,. \end{cases} \qquad (17)$$

Following Williams [18] and others [10–12], the equations (15) and (16) are converted into the following four simultaneous Fredholm integral equations of the second kind:

$$T_1^{(m)}(\rho) = l_2^{(m)}(\rho) + \frac{(m+1)!}{\rho^{m+1}(\Gamma(m+\frac{5}{2}))(\pi)^{\frac{1}{2}}} \times$$
$$\times \int_0^b \frac{u^{m+2}\, T_2^{(m)}(u)\, {}_2F_1(\frac{1}{2}, m+1; m+\frac{5}{2}; u^2/\rho^2)\, du}{(\rho^2-u^2)}\,, \qquad (a<\rho<\infty)\,, \qquad (18)$$

$$T_2^{(m)}(\rho) = l_1^{(m)}(\rho) + \frac{\rho^{m+2}(m+1)!}{(\Gamma(m+\frac{5}{2}))\pi^{\frac{1}{2}}} \times$$
$$\times \int_a^\infty \frac{u^{-m-1}\, T_1^{(m)}(u)\, {}_2F_1(\frac{1}{2}, m+1; m+\frac{5}{2}; \rho^2/u^2)\, du}{(u^2-\rho^2)}\,, \qquad (0<\rho<b)\,, \qquad (19)$$

$$S_1^{(m)}(\rho) + \int_0^a S_1^{(m)}(v)\, L_1^{(m)}(v,\rho)\, dv = C_1^{(m)}(\rho) + \int_a^\infty T_1^{(m)}(v)\, L_1^{(m)}(v,\rho)\, dv\,, \qquad (0<\rho<a)\,, \quad (20)$$

$$S_2^{(m)}(\rho) + \int_b^\infty S_2^{(m)}(v)\, L_2^{(m)}(v,\rho)\, dv = C_2^{(m)}(\rho) + \int_0^b T_2^{(m)}(v)\, L_2^{(m)}(v,\rho)\, dv\,, \qquad (b<\rho<\infty)\,, \quad (21)$$

where

$$\rho^{m+1} \int_\rho^\infty \frac{f_1^{(m)}(t)\, dt}{(t^2-\rho^2)^{\frac{1}{2}}} = \begin{cases} S_1^{(m)}(\rho), & 0<\rho<a\,, \\ -T_1^{(m)}(\rho), & a<\rho<\infty\,, \end{cases} \qquad (22)$$

$$\frac{1}{\rho^{m+1}} \int_0^\rho \frac{t^{2m+2} f_2^{(m)}(t)\, dt}{(\rho^2-t^2)^{\frac{1}{2}}} = \begin{cases} -T_2^{(m)}(\rho), & 0<\rho<b\,, \\ S_2^{(m)}(\rho), & b<\rho<\infty\,, \end{cases} \qquad (23)$$

$$l_1^{(m)}(\rho) = -\frac{2}{\pi\rho^{m+1}} \int_0^\rho \frac{t^{2m+2}}{(\rho^2-t^2)^{\frac{1}{2}}} \frac{d}{dt} \int_t^a \frac{u^{-m} S_1^{(m)}(u)\, du\, dt}{(u^2-t^2)^{\frac{1}{2}}}\,, \qquad (0<\rho<b)\,, \qquad (24)$$

$$l_2^{(m)}(\rho) = \frac{2\rho^{m+1}}{\pi} \int_\rho^\infty \frac{t^{-2m-2}}{(t^2-\rho^2)^{\frac{1}{2}}} \frac{d}{dt} \int_b^t \frac{u^{m+2} S_2^{(m)}(u)\, du\, dt}{(t^2-u^2)^{\frac{1}{2}}}\,, \qquad (a<\rho<\infty)\,, \qquad (25)$$

$$L_1^{(m)}(v,\rho) = L_2^{(m-1)}(v,\rho) = (v\rho)^{\frac{1}{2}} \int_0^\infty (\gamma-p)\, J_{m+\frac{1}{2}}(p\rho)\, J_{m+\frac{1}{2}}(pv)\, dp\,, \qquad (26)$$

$$C_1^{(m)}(\rho) = \frac{1}{\rho^{m+1}} \frac{d}{d\rho} \int_0^\rho \frac{t^{m+2} h_1^{(m)}(t)\, dt}{(\rho^2-t^2)^{\frac{1}{2}}}\,, \qquad (27)$$

$$C_2^{(m)}(\rho) = -\rho^{m+1} \frac{d}{d\rho} \int_\rho^\infty \frac{t^{-m} h_2^{(m)}(t)\, dt}{(t^2-\rho^2)^{\frac{1}{2}}}\,, \qquad (28)$$

and ${}_2F_1$ is a hypergeometric function.

Inverting the integral equations (22) and (23) we obtain

$$f^{(m)}(\rho) = f_1^{(m)}(\rho)+f_2^{(m)}(\rho) = -\frac{2}{\pi}\frac{d}{d\rho}\left[\int_\rho^a \frac{u^{-m}S_1^{(m)}(u)du}{(u^2-\rho^2)^{\frac{1}{2}}}\right.$$

$$\left.-\int_a^\infty \frac{u^{-m}T_1^{(m)}(u)du}{(u^2-\rho^2)^{\frac{1}{2}}}\right] + \frac{2}{\pi\rho^{2m+2}}\frac{d}{d\rho}\left[-\int_0^b \frac{u^{m+2}T_2^{(m)}(u)du}{(\rho^2-u^2)^{\frac{1}{2}}}+\right.$$

$$\left.+\int_b^\rho \frac{u^{m+2}S_2^{(m)}(u)du}{(\rho^2-u^2)^{\frac{1}{2}}}\right], \qquad (b\leqq\rho\leqq a)\,. \tag{29}$$

If we can solve the four equations (18–21) for the unknown functions $S_1^{(m)}(\rho)$, $S_2^{(m)}(\rho)$, $T_1^{(m)}(\rho)$ and $T_2^{(m)}(\rho)$, then we can determine the value of $g^{(m)}(\rho)$ by using the equations (9) and (29). With that aim in view, we first simplify the kernels $L_1^{(m)}$ and $L_2^{(m)}$. Indeed, by using the complex integration method introduced by Noble [14], we have

$$L_1^{(m)}(v,\rho) = L_2^{(m-1)}(v,\rho) = \begin{cases} -i(v\rho)^{\frac{1}{2}}\displaystyle\int_0^k (k^2-p^2)^{\frac{1}{2}} H_{m+\frac{1}{2}}^{(1)}(p\rho)J_{m+\frac{1}{2}}(pv)dp\,, & (\rho\geqq v)\,, \\ -i(v\rho)^{\frac{1}{2}}\displaystyle\int_0^k (k^2-p^2)^{\frac{1}{2}} J_{m+\frac{1}{2}}(p\rho)H_{m+\frac{1}{2}}^{(1)}(pv)dp\,, & (v\geqq\rho)\,, \end{cases} \tag{30}$$

where $H^{(1)}$ is Hankel function of the first kind. The form (30) of the kernels $L_1^{(m)}$ and $L_2^{(m)}$ is useful when k is small as is the case in the present analysis.

3. Far Field Amplitude and Scattering Cross-Section

In terms of spherical polar coordinates (r, θ, φ):

$$\rho = r\sin\theta\,, \qquad z = r\cos\theta\,,$$

the far field amplitude is defined as

$$\Phi(r,\theta,\varphi) = \Phi(\rho,\varphi,z) = -A(\theta,\varphi)\frac{e^{ikr}}{r} + O(r^{-2})\,, \quad \text{as} \quad r\to\infty\,.$$

Comparing it with the integral representation formula for Φ, we easily calculate $A(\theta, \varphi)$ to be

$$A(\theta,\varphi) = \sum_{m=0}^{\infty} A^{(m)}(\theta)\cos m\varphi\,, \tag{31}$$

where

$$A^{(m)}(\theta) = (i)^{-m-1}(k\cos\theta)\int_b^a tg^{(m)}(t)J_m(kt\sin\theta)dt$$

$$= -(i)^{-m-1}\cot\theta\int_b^a \{t^{m+1}J_{m+1}(kt\sin\theta)\}f^{(m)}(t)dt\,, \tag{32}$$

where we have used the edge conditions $g^{(m)}(a) = g^{(m)}(b) = 0$. Now we use the identities

$$J_{m+1}(k\sin\theta t) = t^{-m-1}\left(\frac{2k\sin\theta}{\pi}\right)^{\frac{1}{2}}\int_0^t \frac{J_{m+\frac{1}{2}}(vk\sin\theta)v^{m+\frac{3}{2}}dv}{(t^2-v^2)^{\frac{1}{2}}}$$

$$= t^{m+1}\left(\frac{2k\sin\theta}{\pi}\right)^{\frac{1}{2}}\int_t^\infty \frac{J_{m+\frac{3}{2}}(vk\sin\theta)dv}{v^{m+\frac{1}{2}}(v^2-t^2)^{\frac{1}{2}}}\,, \tag{33}$$

and the relations (22), (23) in the equation (32), and get

$$A^{(m)}(\theta) = -(i)^{-m-1} a \cos\theta \left(\frac{2\alpha}{\pi \sin\theta}\right)^{\frac{1}{2}} \left\{ \int_0^1 S_1^{(m)}(av) v^{\frac{1}{2}} J_{m+\frac{1}{2}}(v\alpha \sin\theta) dv \right.$$
$$- \int_1^\infty T_1^{(m)}(av) v^{\frac{1}{2}} J_{m+\frac{1}{2}}(v\alpha \sin\theta) dv$$
$$- \lambda^{\frac{3}{2}} \int_0^1 T_2^{(m)}(bv) v^{\frac{1}{2}} J_{m+\frac{3}{2}}(v\beta \sin\theta) dv$$
$$\left. + \lambda^{\frac{3}{2}} \int_1^\infty S_2^{(m)}(bv) v^{\frac{1}{2}} J_{m+\frac{3}{2}}(v\beta \sin\theta) dv \right\}, \tag{34}$$

where $\alpha = ka$, $\beta = kb$ and $\lambda = b/a$.

The scattering cross-section σ is defined as the ratio between the average rate at which the acoustic energy is scattered by the annular disk and the average rate at which the acoustic energy of the incident wave crosses a unit area normal to its direction of propagation. For the case of a general plane wave

$$u_0(\rho, \varphi, z) = \exp\{ik(z\cos\theta_0 + r\sin\theta_0\cos\varphi)\},$$

for $0 \leqq \theta_0 < \pi/2$, where θ_0 is the angle of incidence, the value of the scattering cross section is [10]

$$\sigma = 2\pi \sum_{m=0}^{\infty} \frac{1}{\varepsilon_m} \int_0^\pi |A^{(m)}(\theta)|^2 \sin\theta \, d\theta, \tag{35}$$

where $\varepsilon_m = 2 - \delta_{0m}$, and δ_{ij} is the Kronecker delta.

4. Edge Conditions

One readily deduces from the equations (17), (22) and (23) that

$$\frac{2}{\pi}\frac{d}{d\rho}\left[\int_\rho^\infty \frac{u^{-m} T_1^{(m)}(u) du}{(u^2-\rho^2)^{\frac{1}{2}}}\right] + \frac{2}{\pi\rho^{2m+2}}\frac{d}{d\rho}\left[-\int_0^b \frac{u^{m+2} T_2^{(m)}(u) du}{(\rho^2-u^2)^{\frac{1}{2}}}\right.$$
$$\left. + \int_b^\rho \frac{u^{m+2} S_2^{(m)}(u) du}{(\rho^2-u^2)^{\frac{1}{2}}}\right] = 0, \qquad (a < \rho < \infty). \tag{36}$$

If the function $T_1^{(m)}(\rho)$ is extended over the range $b < \rho \leqq a$, the equation (36) will also hold for the range $b < \rho \leqq a$. But then the equation (29) yields

$$f^{(m)}(\rho) = -\frac{2}{\pi}\frac{d}{d\rho}\left[\int_\rho^a \frac{u^{-m}(S_1^{(m)}(u) + T_1^{(m)}(u)) du}{(u^2-\rho^2)^{\frac{1}{2}}}\right], \qquad (b < \rho \leqq a). \tag{37}$$

Similarly

$$-\frac{2}{\pi}\frac{d}{d\rho}\left[\int_\rho^a \frac{u^{-m} S_1^{(m)}(u) du}{(u^2-\rho^2)^{\frac{1}{2}}} - \int_a^\infty \frac{u^{-m} T_1^{(m)}(u) du}{(u^2-\rho^2)^{\frac{1}{2}}}\right]$$
$$+ \frac{2}{\pi\rho^{2m+2}}\frac{d}{d\rho}\left[-\int_0^\rho \frac{u^{m+2} T_2^{(m)}(u) du}{(\rho^2-u^2)^{\frac{1}{2}}}\right] = 0, \qquad (0 \leqq \rho < b), \tag{38}$$

and it will also hold for $b \leqq \rho < a$ if the function $T_2^{(m)}(\rho)$ is extended over the range $b \leqq \rho < a$. Then the relation (29) will become

$$f^{(m)}(\rho) = \frac{2}{\pi\rho^{2m+2}}\frac{d}{d\rho}\left[\int_b^\rho \frac{u^{m+2}\{S_2^{(m)}(u) + T_2^{(m)}(u)\} du}{(\rho^2-u^2)^{\frac{1}{2}}}\right], \qquad (b \leqq \rho < a). \tag{39}$$

The equations (37) and (39) yield, after some manipulations, the interesting results:

$$f^{(m)}(\rho) = \frac{2}{\pi a^m}\left[\frac{S_1^{(m)}(a) + T_1^{(m)}(a)}{(a^2-\rho^2)^{\frac{1}{2}}} + O\{(a^2-\rho^2)^{\frac{1}{2}}\}\right] \quad \text{as} \quad \rho \to a, \tag{40}$$

and

$$f^{(m)}(\rho) = \frac{2}{\pi b^m}\left[\frac{S_2^{(m)}(b)+T_2^{(m)}(b)}{(\rho^2-b^2)^{\frac{1}{2}}} + O\{(\rho^2-b^2)^{\frac{1}{2}}\}\right] \quad \text{as} \quad \rho\to b\,. \tag{41}$$

The relations (40) and (41) constitute the edge conditions for the rigid annular disk. When $\lambda\to 0$, the relations (40) and (41) reduce to the known results for the rigid circular disk.

5. Special Case of an Axially Symmetric Plane Wave

Details of the above analysis can be presented when the incident wave is a plane wave travelling in the direction of the positive z-axis, i.e.

$$u_0(\rho,\varphi,z)=e^{ikz},\quad u_0^{(0)'}(\rho)=ik,\quad u_0^{(m)}(\rho,z)=0,\qquad m\geqq 1\,.$$

Then it follows from the relations (12–14) that

$$h_1^{(0)}(\rho)=ik\rho/2,\qquad h_2^{(0)}(\rho)=0\,.$$

For this special case the simultaneous system of governing integral equations are the equations (18–21) with m set equal to zero:

$$T_1^{(0)}(a\rho)=l_2^{(0)}(a\rho)+\frac{1}{\pi}\int_0^1 T_2^{(0)}(bu)\left\{\frac{2\lambda\rho}{(\rho^2-\lambda^2u^2)}-\frac{1}{u}\log\left(\frac{\rho+\lambda u}{\rho-\lambda u}\right)\right\}du,\ (1<\rho<\infty)\,, \tag{42}$$

$$T_2^{(0)}(b\rho)=l_1^{(0)}(b\rho)+\frac{1}{\lambda\rho\pi}\int_1^\infty T_1^{(0)}(au)\left\{\frac{2\lambda u\rho}{(u^2-\lambda^2\rho^2)}-\log\left(\frac{u+\lambda\rho}{u-\lambda\rho}\right)\right\}du,\ (0<\rho<1)\,, \tag{43}$$

$$S_1^{(0)}(a\rho)=i\alpha\rho+a\int_1^\infty T_1^{(0)}(av)L_1^{(0)}(av,a\rho)dv-a\int_0^1 S_1^{(0)}(av)L_1^{(0)}(av,a\rho)dv,\ (0<\rho<1), \tag{44}$$

$$S_2^{(0)}(b\rho)=b\int_0^1 T_2^{(0)}(bv)L_2^{(0)}(bv,b\rho)dv-b\int_1^\infty S_2^{(0)}(bv)L_2^{(0)}(bv,b\rho)dv,\qquad (1<\rho<\infty), \tag{45}$$

where $\lambda=b/a$. Let us collect the small perturbation parameters occurring in this analysis:

$$\alpha=ka,\quad \beta=kb,\quad \lambda=b/a=\beta/\alpha$$

and we shall assume in the sequel that $\alpha=O(\lambda)$ and that $\beta=\alpha\lambda=O(\alpha^2)$. Fortunately, the kernels of the equations (42–45) tend to zero as λ, α, and β tend to zero, i.e. when the frequency of the incident wave is low and when the inner radius of the annular disk is small. We can easily give the expansions of these kernels in terms of these parameters, since the kernels of all these equations are given in terms of elementary functions. Indeed, from (30) we have

$$L_1^{(0)}(v,\rho)=\begin{cases} -i(v\rho)^{\frac{1}{2}}\int_0^k (k^2-p^2)^{\frac{1}{2}}H_{\frac{1}{2}}^{(1)}(p\rho)J_{\frac{1}{2}}(pv)dp\,, & \rho\geqq v\,,\\ -i(v\rho)^{\frac{1}{2}}\int_0^k (k^2-p^2)^{\frac{1}{2}}J_{\frac{1}{2}}(p\rho)H_{\frac{1}{2}}^{(1)}(pv)dp\,, & v\geqq\rho\,,\end{cases}$$

$$=\begin{cases} -\frac{k^2v}{2}-\frac{2i\rho vk^3}{3\pi}+\frac{(v^3+3\rho^2v)k^4}{48}+\frac{2i(\rho^3v+\rho v^3)k^5}{45\pi}-\frac{(v^5+10\rho^2v^3+5\rho^4v)k^6}{1920}\\ \qquad -\frac{2i(3\rho v^5+10\rho^3v^3+3\rho^5v)k^7}{4725\pi}+O(k^8)\,, \qquad \rho\geqq v\,,\\ -\frac{k^2\rho}{2}-\frac{2i\rho vk^3}{3\pi}+\frac{(\rho^3+3\rho v^2)k^4}{48}+\frac{2i(\rho v^3+\rho^3v)k^5}{45\pi}-\frac{(\rho^5+10v^2\rho^3+5v^4\rho)k^6}{1920}\\ \qquad -\frac{2i(3\rho v^5+10\rho^3v^3+3\rho^5v)k^7}{4725\pi}+O(k^8)\,, \qquad v\geqq\rho\,.\end{cases} \tag{46}$$

Similarly

$$L_2^{(0)}(v,\rho)=\begin{cases}-\dfrac{k^2v^2}{6\rho}+O(k^3)\,, & \rho\geqq v\,,\\[2ex] -\dfrac{k^2\rho^2}{6v}+O(k^3)\,, & v\geqq\rho\,,\end{cases}\tag{47}$$

Let us set

$$S_1^{(0)}(a\rho)=X_1^{(0)}(a\rho)+W_1^{(0)}(a\rho)\,,$$

in the equation (44) which can then be split into the following two equations

$$X_1^{(0)}(a\rho)=i\alpha\rho-a\int_0^1 X_1^{(0)}(av)\,L_1^{(0)}(av,a\rho)\,dv \qquad (0<\rho<1)\,,\tag{48}$$

and

$$W_1^{(0)}(a\rho)=a\int_1^\infty T_1^{(0)}(av)\,L_1^{(0)}(av,a\rho)dv-a\int_0^1 L_1^{(0)}(av,a\rho)\,W_1^{(0)}(av)dv \qquad (0<\rho<1)\,.\tag{49}$$

The equation (48) is the integral equation for the problem of diffraction of an axially symmetric acoustic plane wave by a perfectly rigid disk as solved by Williams [13]. However, Williams did not split the kernel as we have done above and therefore his analysis cannot be used. But this equation is a simple Fredholm integral equation of the second kind and can be readily solved by iteration to obtain approximate value of $X_1^{(0)}(a\rho)$:

$$X_1^{(0)}(a\rho)=i\alpha\left[c_1(\alpha)\rho+c_3(\alpha)\rho^3+c_5(\alpha)\rho^5+c_7(\alpha)\rho^7+O(\alpha^8)\right],\tag{50}$$

where

$$c_1(\alpha)=1+\frac{\alpha^2}{4}+\frac{2i\alpha^3}{9\pi}+\frac{7\alpha^4}{192}+\frac{41i\alpha^5}{450\pi}+\left(\frac{11}{3840}-\frac{4}{81\pi^2}\right)\alpha^6+\frac{6301i\alpha^7}{352800\pi}\,,$$

$$c_3(\alpha)=-\frac{\alpha^2}{12}-\frac{\alpha^4}{32}-\frac{i\alpha^5}{30\pi}-\frac{\alpha^6}{256}-\frac{101i\alpha^7}{8400\pi}\,,$$

$$c_5(\alpha)=\frac{\alpha^4}{320}+\frac{\alpha^6}{768}+\frac{i\alpha^7}{672\pi}\,,$$

$$c_7(\alpha)=-\frac{\alpha^6}{16128}\,.$$

Having found $X_1^{(0)}$, we can calculate the other functions in the following sequence:

$$X_1^{(0)},\quad l_1^{(0)},\quad T_2^{(0)},\quad S_2^{(0)},\quad l_2^{(0)},\quad T_1^{(0)},\quad W_1^{(0)},\quad S_1^{(0)}\,,$$

by iteration and the results are

$$l_1^{(0)}(b\rho)=\frac{4i\alpha\lambda^2}{3\pi}\Bigg[\rho^2\left\{c_1(\alpha)-c_3(\alpha)-\tfrac{1}{3}c_5(\alpha)-\tfrac{1}{5}c_7(\alpha)\right\} +\frac{2\lambda^2\rho^4}{5}\left(1+\frac{2i\alpha^3}{9\pi}-\frac{\alpha^4}{15}\right)+\frac{9\lambda^4\rho^6}{35}\left(1+\frac{\alpha^2}{9}\right)+\frac{4\lambda^6\rho^8}{21}+O(\alpha^7)\Bigg],$$

$$(0<\rho<1)\,,\tag{51}$$

$$T_2^{(0)}(b\rho)=l_1^{(0)}(b\rho)+\frac{64i\alpha\lambda^7\rho^2}{675\pi^3}\left[1+O(\alpha^2)\right], \qquad (0<\rho<1)\,,\tag{52}$$

$$S_2^{(0)}(b\rho)=-\frac{2i\alpha^3\lambda^4}{45\pi}\left[\frac{1}{\rho}+O(\alpha^2)\right], \qquad (1<\rho<\infty)\,,\tag{53}$$

$$I_2^{(0)}(a\rho) = -\frac{4i\alpha^3\lambda^5}{45\pi^2}\left\{\frac{1}{\rho} + O(\alpha^2)\right\}, \qquad (1<\rho<\infty), \tag{54}$$

$$T_1^{(0)}(a\rho) = \frac{16i\alpha\lambda^5}{45\pi^2}\left[-\frac{\alpha^2}{4\rho} + \frac{1}{\rho^3}\left(1 + \frac{\alpha^3}{3} + \frac{2\lambda^2}{7}\right) + \frac{6\lambda^2}{7\rho^5} + O(\alpha^3)\right], \quad (1<\rho<\infty), \tag{55}$$

$$W_1^{(0)}(a\rho) = -\frac{4i\alpha^3\lambda^5}{45\pi^2}\left[\rho + O(\alpha)\right], \qquad (0<\rho<1), \tag{56}$$

and

$$S_1^{(0)}(a\rho) = X_1^{(0)}(a\rho) - \frac{4i\alpha^3\lambda^5\rho}{45\pi^2} + O(\alpha^9), \qquad (0<\rho<1), \tag{57}$$

We can find the behaviour of the diffracted field at infinity for this special case by appealing to the formula (34) with m set equal to zero:

$$\begin{aligned} A^{(0)}(\theta) = ia\cos\theta\left(\frac{2\alpha}{\pi\sin\theta}\right)^{\frac{1}{2}}\Bigg[&\int_0^1 S_1^{(0)}(av)v^{\frac{1}{2}}J_{\frac{1}{2}}(\alpha v\sin\theta)dv \\ &-\int_1^\infty T_1^{(0)}(av)v^{\frac{1}{2}}J_{\frac{1}{2}}(\alpha v\sin\theta)dv \\ &-\lambda^{\frac{3}{2}}\int_0^1 T_2^{(0)}(bv)v^{\frac{1}{2}}J_{\frac{3}{2}}(\beta v\sin\theta)dv \\ &+\lambda^{\frac{3}{2}}\int_1^\infty S_2^{(0)}(bv)v^{\frac{1}{2}}J_{\frac{3}{2}}(\beta v\sin\theta)dv\Bigg]. \end{aligned} \tag{58}$$

When we use the approximations of the functions $S_1^{(0)}$, $S_2^{(0)}$, $T_1^{(0)}$ and $T_2^{(0)}$ as given by the relations (57), (53), (55), and (52) respectively as well as the values of certain well known infinite integrals involving Bessel functions, there results

$$\begin{aligned} A^{(0)}(\theta) = -\frac{2a\cos\theta\alpha^2}{3\pi}\Bigg[&\Bigg\{1 + \frac{\alpha^2}{5} + \frac{2i\alpha^3}{9\pi} + \frac{2\alpha^4}{105} + \frac{16i\alpha^5}{225\pi} + \left(\frac{1}{945} - \frac{4}{81\pi^2}\right)\alpha^6 \\ &-\frac{\alpha^2}{10}\left(1 + \frac{4\alpha^2}{21} + \frac{2i\alpha^3}{9\pi} + \frac{\alpha^4}{63}\right)\sin^2\theta + \left(1 + \frac{5\alpha^2}{27}\right)\frac{\alpha^4}{280}\sin^4\theta \\ &-\frac{\alpha^6}{15120}\sin^6\theta + O(\alpha^7)\Bigg\} - \frac{16\lambda^5}{15\pi^2}\left\{1 + \frac{2\alpha^2}{3} + \frac{4\lambda^2}{7} + \frac{\alpha^2\sin^2\theta}{6} + O(\alpha^3)\right\}\Bigg]. \end{aligned} \tag{59}$$

From the relations (35) and (59) we finally obtain an approximation to the plane wave scattering cross-section σ:

$$\begin{aligned} \sigma = \frac{16a^2\alpha^4}{27\pi}\Bigg[&1 + \frac{8\alpha^2}{25} + \frac{311\alpha^4}{6125} + \left(\frac{2612}{496125} - \frac{4}{81\pi^2}\right)\alpha^6 \\ &-\frac{32\lambda^5}{15\pi^2} - \frac{128\lambda^7}{105\pi^2} - \frac{2144\lambda^5\alpha^2}{1125\pi^2} + O(\alpha^8)\Bigg]. \end{aligned} \tag{60}$$

When $\lambda\to 0$, the relations (59) and (60) reduce to the well known results for the rigid circular disk of radius a, [15], [16].

As regards the edge conditions, which were explained in section 4, we have verified that the solution $f^{(0)}(\rho)$ in this special case satisfies the formulae (40) and (41) as well as (37) and (38).

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